Elliptic Ruijsenaars - Schneider model via the Poisson reduction of the affine Heisenberg double

1997 J. Phys. A: Math. Gen. 30 5051

(http://iopscience.iop.org/0305-4470/30/14/016)

J. Phys. A: Math. Gen. **30** (1997) 5051–5063. Printed in the UK PII: S0305-4470(97)79196-0

Elliptic Ruijsenaars–Schneider model via the Poisson reduction of the affine Heisenberg double

G E Arutyunov†§, S A Frolov†‖ and P B Medvedev‡

† Steklov Mathematical Institute, Vavilov 42, GSP-1, 117966, Moscow, Russia

‡ Institute of Theoretical and Experimental Physics, B Cheremushkinskaja 25, 117259 Moscow, Russia

Received 4 November 1996

Abstract. It is shown that the elliptic Ruijsenaars–Schneider model can be obtained from the affine Heisenberg double by means of the Poisson reduction procedure. The dynamical r-matrix naturally appears in the construction.

1. Introduction

The recent development [1–7] in the theory of integrable many-body systems is mainly related with the discovery [9] of dynamical r-matrices, i.e. r-matrices depending on phase variables. One natural way to understand the origin of dynamical r-matrices is to consider the reduction procedure [10–14, 2, 6]. In this approach one starts with an initial phase space $\mathcal{P}$ supplied with a symplectic action of some symmetry group. Considering a relatively simple invariant Hamiltonian and factorizing the corresponding dynamics by the symmetry group one obtains a smaller phase space $\mathcal{P}_{\mathrm{red}}$ with a nontrivial dynamics. Then the L-operator coming in the Lax representation $\mathrm{d}L/\mathrm{d}t = [M, L]$ appears as a specific coordinate on $\mathcal{P}_{\mathrm{red}}$ while the dynamical r-matrix describes the Poisson (Dirac) bracket on the reduced space.

At present the reduction procedure has been elaborated for the majority of integrable many-body systems and the corresponding r-matrices have been derived. One of the most interesting exceptions is the elliptic Ruijsenaars–Schneider model [15]. Recently, two different dynamical r-matrices for this model were found in [7] and [8]. Both of these r-matrices were obtained by a direct calculation and the question of their equivalence still remains open.

In this paper we apply the Poisson reduction procedure to the affine Heisenberg double (HD) [16] and derive the elliptic Ruijsenaars–Schneider model with the dynamical r-matrix. The reason for using the affine HD becomes apparent due to its relation to integrable many-body systems of Calogero type. As was shown in [17, 18] the Calogero–Moser and the rational and trigonometric Ruijsenaars–Schneider hierarchies can be obtained by means of the reduction procedure from the cotangent bundle of an affine Lie group $T^*G(z)$ and from a finite dimensional Heisenberg double. The affine Heisenberg double may be regarded as a deformation of $T^*G(z)$ and therefore one can suggest that the affine HD is a natural candidate for the phase space standing behind the elliptic Ruijsenaars–Schneider system.

§ E-mail address: arut@class.mi.ras.ru

‖ E-mail address: frolov@class.mi.ras.ru

0305-4470/97/145051+13$19.50

The plan of the paper is as follows. In section 2 we briefly describe the affine HD in terms of variables which are suitable for the reduction procedure. Then we fix the momentum map, corresponding to the natural action of the affine Poisson–Lie group on the HD. The solution of the momentum map equation is shown to be equivalent to the L-operator of the elliptic Ruijsenaars–Schneider model. In section 3 we study the Poisson structure of the reduced phase space and prove that it coincides with that of the elliptic Ruijsenaars–Schneider model. The dynamical r-matrix naturally appears in our consideration and is equivalent to that obtained in [7]. In section 4 we show that the problem of solving the equations of motion is equivalent to the specific factorization problem. We give brief conclusions in section 5.

In our presentation we omit the detailed description of the HD and the proof of some statements. A complete discussion will be given in a forthcoming publication.

2. Affine Heisenberg double

The general construction of a Poisson manifold known as the Heisenberg double was elaborated in [16]. We shall discuss the HD for affine $\widehat{GL(N)}$. It is convenient to describe the Poisson structure of the affine HD in the following form. Let $A(x)$ and $C(x)$ be formal Fourier series in a variable x with values in $GL(N)$. The matrix elements of the harmonics of $A(x)$ and $C(x)$ can be regarded as generators of the algebra of functions on the HD. The Poisson structure on the HD appears as follows:

$$\begin{aligned}
\frac{1}{\gamma}\{A_1(x), A_2(y)\} &= -r_{\mp}(x-y)A_1(x)A_2(y) - A_1(x)A_2(y)r_{\pm}(x-y) \\
&\quad + A_2(y)r_+(x-y-2\Delta)A_1(x) + A_1(x)r_-(x-y+2\Delta)A_2(y) \\
\frac{1}{\gamma}\{C_1(x), C_2(y)\} &= -r_{\mp}(x-y)C_1(x)C_2(y) - C_1(x)C_2(y)r_{\pm}(x-y) \\
&\quad + C_2(y)r_+(x-y)C_1(x) + C_1(x)r_-(x-y)C_2(y) \\
\frac{1}{\gamma}\{A_1(x), C_2(y)\} &= -r_-(x-y)A_1(x)C_2(y) - A_1(x)C_2(y)r_-(x-y+2\Delta) \\
&\quad + C_2(y)r_+(x-y)A_1(x) + A_1(x)r_-(x-y+2\Delta)C_2(y) \\
\frac{1}{\gamma}\{C_1(x), A_2(y)\} &= -r_+(x-y)C_1(x)A_2(y) - C_1(x)A_2(y)r_+(x-y-2\Delta) \\
&\quad + A_2(y)r_+(x-y-2\Delta)C_1(x) + C_1(x)r_-(x-y)A_2(y)
\end{aligned}$$

where γ and Δ are complex numbers with $\mathrm{Im}\,\Delta > 0$. Here we use a standard tensor notation. The matrices $r_\pm(x)$ are defined by their Fourier series as

$$r_+(x) = r_+ + P\sum_{n>0} e^{-\mathrm{i}nx} \qquad r_-(x) = r_- - P\sum_{n>0} e^{\mathrm{i}nx}$$

where

$$r_+ = \frac{1}{2}\sum_i E_{ii}\otimes E_{ii} + \sum_{i<j} E_{ij}\otimes E_{ji}$$

and $r_- = -Pr_+P$ and P is the permutation operator. It can easily be checked that

$$r_+(x) - r_-(x) = 2\pi P\delta(x) \qquad \text{and} \qquad Pr_+(x)P = -r_-(-x).$$

In the region of convergence the $r_{\pm}(x)$ coincide with the standard trigonometric r-matrix for the affine Lie algebra. The Poisson subalgebra generated by $A(x)$ was introduced in [19] to describe the Poisson structure of $\widehat{GL(N)}^*$.

Assuming the expansions

$$A(x) = I + \gamma J(x) + \ldots \qquad C(x) = g(x) + \ldots \qquad 2\Delta = \gamma k$$

where k is a (fixed) central charge, in the deformation limit $\gamma \to 0$ we recover the standard Poisson structure on the cotangent bundle $T^*\widehat{GL(N)}$ over the level-k, centrally extended current group $\widehat{GL(N)}$.

The action of the current group $\widetilde{GL(N)}$ on the HD:

$$A(x) \to T^{-1}(x-\Delta)A(x)T(x+\Delta)$$

$$C(x) \to T^{-1}(x-\Delta)C(x)T(x-\Delta)$$

is the Poisson one. We can thereby consider the Poisson reduction of HD over the action of $\widetilde{GL(N)}$.

The momentum map taking value in $\widetilde{GL(N)}^*$ reads as follows:

$$M(x) = A^{-1}(x-\Delta)C(x-\Delta)A(x-\Delta)C^{-1}(x+\Delta).$$

It is easy to check that $M(x)$ does generate the action of the current group. We fix the value of $M(x)$ as:

$$M(x) = e^{\mathrm{i}h}\left(1 - 2\pi\mathrm{i}\delta_\varepsilon(x)\frac{1-e^{-\mathrm{i}x}}{\mathrm{i}}K\right). \tag{2.1}$$

Here h and ε are arbitrary complex numbers

$$\delta_\varepsilon(x) = \frac{1}{\varepsilon}\left(\theta(x+\tfrac{1}{2}\varepsilon) - \theta(x-\tfrac{1}{2}\varepsilon)\right) = \frac{1}{2\pi\mathrm{i}\varepsilon}\sum_{n=-\infty}^{\infty}\frac{1}{n}(e^{\mathrm{i}n\frac{\varepsilon}{2}} - e^{-\mathrm{i}n\frac{\varepsilon}{2}})e^{\mathrm{i}nx}$$

and K is a constant matrix $K = e \otimes e^t$, where e is the N-dimensional vector with entries $e_i = 1/\sqrt{N}$†

Although equation (2.1) can be solved for any value of ε, the reduced phase space remains to be infinite dimensional after performing the factorization procedure. To extract a finite-dimensional phase space let us carry out the following trick. By multiplying the both sides of (2.1) on $C(x+\Delta)$, one obtains

$$C(x+\Delta) - e^{-\mathrm{i}h}A^{-1}(x-\Delta)C(x-\Delta)A(x-\Delta) = 2\pi\mathrm{i}\delta_\varepsilon(x)K\frac{1-e^{-\mathrm{i}x}}{\mathrm{i}}C(x+\Delta). \tag{2.2}$$

The left-hand side of this equation does not have any explicit dependence on ε. As for the right-hand side, when ε tends to zero, $\delta_\varepsilon(x)$ tends to $\delta(x)$ and the right-hand side is well defined only if the function $((1-e^{-\mathrm{i}x})/\mathrm{i})C(x+\Delta)$ is well defined at $x=0$. Hence, this equation can be solved only for meromorphic functions $C(x+\Delta)$ with poles of the first order. In this case $\lim_{\varepsilon\to 0}\delta_\varepsilon(x)((1-e^{-\mathrm{i}x})/\mathrm{i})C(x+\Delta) = \delta(x)\,\mathrm{Res}_{x=0}\,C(x+\Delta)$.

So we define the constraint surface as being the solution of the equation

$$C(x+\Delta) - e^{-\mathrm{i}h}A^{-1}(x-\Delta)C(x-\Delta)A(x-\Delta) = 2\pi\mathrm{i}\delta(x)K\,\mathrm{Res}_{x=0}\,C(x+\Delta)$$

and in what follows we shall explore solutions of this equation.

† It is worth noting that in the deformation limit $\gamma \to 0$, $h/\gamma \to$ constant, $\varepsilon/\gamma \to$ constant the constraint (2.1) reduces to that used in [17] to obtain the trigonometric Ruijsenaars model.

We start with the following difference equation:

$$C(x+\Delta) - e^{-\mathrm{i}h} D^{-1} C(x-\Delta) D = 2\pi \mathrm{i} \delta(x) Y \tag{2.3}$$

where D is a constant diagonal matrix and Y is an arbitrary constant matrix. Performing the Fourier expansion we obtain a solution of (2.3) in the form

$$C(x) = \mathrm{i} \sum_{ij} \sum_{n=-\infty}^{\infty} \frac{e^{\mathrm{i}nx}}{e^{\mathrm{i}n\Delta} - e^{-\mathrm{i}s_{ij}} e^{-\mathrm{i}n\Delta}} Y_{ij} E_{ij}$$

where we use the notation $s_{ij} = h + q_{ij}$, $D = e^{\mathrm{i}q}$, $q_{ij} = q_i - q_j$. It is useful to introduce the function of two complex variables

$$w(x,s) = \mathrm{i} \sum_{n=-\infty}^{\infty} \frac{e^{\mathrm{i}nx}}{e^{\mathrm{i}n\Delta} - e^{-\mathrm{i}s} e^{-\mathrm{i}n\Delta}}.$$

It is clear that $w(x,s)$ is a meromorphic function of s for any $x : |\operatorname{Im} x| < \operatorname{Im} \Delta$ and has two obvious properties

(i) $w(x, s+2\pi) = w(x,s)$,
(ii) $w(x, s+2\Delta) = e^{\mathrm{i}\Delta - \mathrm{i}x} w(x,s)$.

Moreover, as a function of s it has simple poles at $0, \pm 2\pi, \pm 4\pi, \ldots$ and $\pm 2\Delta, \pm 4\Delta, \ldots$, $\operatorname{Res}_{s=0} w = 1$. By these data w is uniquely defined as

$$w(x,s) = \frac{\sigma(s+x-\Delta)}{\sigma(x-\Delta)\sigma(s)} \exp\left(-\frac{\zeta(\pi)}{\pi}(x-\Delta)s\right). \tag{2.4}$$

Here $\sigma(x)$ and $\zeta(x)$ are the Weierstrass σ- and ζ-functions with periods equal to 2π and 2Δ. Thus, equation (2.3) has the unique solution

$$\begin{aligned} C(x) &= \sum_{ij} \frac{\sigma(q_{ij}+h+x-\Delta)}{\sigma(x-\Delta)\sigma(q_{ij}+h)} \exp\left(-\frac{\zeta(\pi)}{\pi}(x-\Delta)(h+q_{ij})\right) Y_{ij} E_{ij} \\ &= \sum_{ij} w(x, s_{ij}) Y_{ij} E_{ij}. \end{aligned} \tag{2.5}$$

Now we turn to the momentum map equation

$$C(x+\Delta) - e^{-\mathrm{i}h} A^{-1}(x-\Delta) C(x-\Delta) A(x-\Delta) = 2\pi \mathrm{i} K Z \delta(x) \tag{2.6}$$

where $Z = \operatorname{Res}_{x=0} C(x+\Delta)$.

By using a generic gauge transformation we can diagonalize the field A. Then equation (2.6) takes the form of (2.3)

$$C'(x+\Delta) - e^{-\mathrm{i}h} D^{-1} C'(x-\Delta) D = 2\pi \mathrm{i} K' Z' \delta(x) \tag{2.7}$$

where

$$A(x) = T(x-\Delta) D T^{-1}(x+\Delta) \qquad C(x) = T(x-\Delta) C'(x) T^{-1}(x-\Delta)$$

for some T and $Z' = \operatorname{Res}_{x=\Delta} C'(x)$. We also have

$$K' = T^{-1}(0) K T(0) = T^{-1}(0) e \otimes e^t T(0) = f \otimes v^t \qquad \langle f, v \rangle = 1$$

i.e. $f = T^{-1}(0)e$ and $e^t T(0) = v^t$. According to (2.5) we find

$$C'(x) = \sum_{ij} w(x, s_{ij}) (K'Z')_{ij} E_{ij}.$$

Taking the residue of $C'(x)$ at the point $x = \Delta$ we arrive at the compatibility condition

$$Z' = K'Z' = f \otimes v^t Z' \qquad \langle f, v \rangle = 1.$$

The solution of this equation is $Z' = f \otimes g^t$, where g is an arbitrary vector. Now it is easy to find Z:

$$Z = T(0)Z'T^{-1}(0) = T(0)f \otimes g^t T^{-1}(0) = e \otimes g^t T^{-1}(0) \equiv e \otimes b^t.$$

Thus, we obtain

$$C(x+\Delta) - e^{-\mathrm{i}h} A^{-1}(x-\Delta)C(x-\Delta)A(x-\Delta) = 2\pi\mathrm{i}(e \otimes e^t)(e \otimes b^t)\delta(x) \tag{2.8}$$

where $e \otimes b^t$ is a residue of $C(x)$ at $x = \Delta$.

In summary, equation (2.6) has a solution for any field A and for any field C, having a residue at $x = \Delta$ of the form $e \otimes b^t$. For a fixed field A and a vector b this solution is unique. Note that, in general, $\langle b, e\rangle \neq 1$. The form of the right-hand side of (2.8) shows that the isotropy group of this equation is

$$G_{\mathrm{isot}} = \{T(x) \subset G(x) \ \mid \ T(0)e = \lambda e, \lambda \in \mathbb{C}\}.$$

This group transforms a solution of (2.8) into another one, so the reduced phase space is defined as

$$\mathcal{P}_{\mathrm{red}} = \frac{\text{all solutions of (2.6)}}{G_{\mathrm{isot}}}.$$

Since the group G_{isot} is large enough to diagonalize the field A, we can parametrize the reduced phase space by the section (D, L), where L is a solution of (2.6) with $A = D$. One can easily see that $\mathcal{P}_{\mathrm{red}}$ is finite dimensional and its dimension is equal to $2N$, i.e. N coordinates of D plus N coordinates of the vector b. Due to equation (2.5) the corresponding L-operator has the following form:

$$L(x) = \sum_{ij} \frac{\sigma(q_{ij} + h + x - \Delta)}{\sigma(x-\Delta)\sigma(q_{ij} + h)} \exp\left(-\frac{\zeta(\pi)}{\pi}(x-\Delta)(h + q_{ij})\right) e_i b_j E_{ij}. \tag{2.9}$$

Multiplying $L(x)$ by the function $(\sigma(x-\Delta)\sigma(h)/\sigma(x-\Delta+h))e^{\zeta(\pi)(x-\Delta)h/\pi}$, performing the gauge transformation by means of the diagonal matrix $e^{\zeta(\pi)(x-\Delta)q/\pi}$, and making the shift $x \to x + \Delta$ we obtain the L-operator of the elliptic Ruijsenaars–Schneider model:

$$L^{\mathrm{Ruij}}(x) = \frac{\sigma(x)\sigma(h)}{\sigma(x+h)} e^{\zeta(\pi)xh/\pi} e^{\zeta(\pi)xq/\pi} L(x+\Delta) e^{-\zeta(\pi)xq/\pi}. \tag{2.10}$$

Let us briefly discuss the Hamiltonian. It is well known that the simplest non-trivial Hamiltonian invariant with respect to the action of the current group is given by:

$$H = \int \mathrm{d}x \, \mathrm{tr}\, C(x) \tag{2.11}$$

where α is a constant. It is not difficult to show that on the reduced phase space

$$H_{\mathrm{red}} = \frac{2\pi\mathrm{i}}{\sqrt{N}(1 - e^{-\mathrm{i}h})} \sum_{i=1}^{N} b_i \tag{2.12}$$

that, up to a constant, is the simplest Hamiltonian of the elliptic Ruijsenaars–Schneider model.

3. The Poisson structure on the reduced space

In this section we are going to prove that the Poisson structure on the reduced phase space does indeed coincide with the Poisson structure of the elliptic Ruijsenaars–Schneider model. In other words we need Poisson brackets for the coordinates D and b. According to the general Dirac construction one should find a gauge invariant extension (we mean the invariance under the action of G_{isot}) of functions on the reduced phase space $\mathcal{P}_{\mathrm{red}}$ to a vicinity of $\mathcal{P}_{\mathrm{red}}$ and then calculate the Dirac bracket.

One can easily write down the gauge invariant extension for the matrices D and $L(x)$ while the bracket for the coordinates D_i and b_i can be extracted from the bracket for D and $L(x)$. This extension appears as follows:

$$D \to D[A] = T^{-1}[A](x-\Delta)A(x)T[A](x+\Delta) \tag{3.1}$$

$$L(x) \to \mathcal{L}[A,C](x) = T^{-1}[A](x-\Delta)C(x)T[A](x-\Delta). \tag{3.2}$$

Some comments are in order. Equation (3.1) is a solution of the factorization problem for $A(x)$. Generally this solution is not unique but we fix the matrix $T[A]$ by the boundary condition $T[A](0)e = e$ that kills the ambiguity and makes (3.1) to be correctly defined. It is obvious that on $\mathcal{P}_{\mathrm{red}}$: $T[A] = 1$ and $\mathcal{L}[A,C](x) = L(x)$.

We start with calculation of the Poisson bracket for $\mathcal{L}(x)$ and $\mathcal{L}(y)$. We postpone discussion of the contribution from the second-class constraints to the Dirac bracket to the end of the section. By definition, one has

$$\begin{aligned}\{\mathcal{L}_1,\mathcal{L}_2\}_{\mathcal{P}_{\mathrm{red}}} = (&\{T_1,T_2\}L_1L_2 - L_2\{T_1,T_2\}L_1 - L_1\{T_1,T_2\}L_2 \\ &+ L_1L_2\{T_1,T_2\} + \{C_1,C_2\} - \{T_1,C_2\}L_1 - \{C_1,T_2\}L_2 \\ &+ L_2\{C_1,T_2\} + L_1\{T_1,C_2\})\,|_{\mathcal{P}_{\mathrm{red}}}.\end{aligned} \tag{3.3}$$

Here we have taken into account that $T[A]|_{\mathcal{P}_{\mathrm{red}}} = 1$.

Let us first calculate

$$\{C_{ij}(x), T_{kl}(y)\} = \sum_{m,n}\int \mathrm{d}z\ \{C_{ij}(x), A_{mn}(z)\}\frac{\delta T_{kl}(y)}{\delta A_{mn}(z)}.$$

Performing the variation of both sides of (3.1), we obtain

$$X(x) = t(x-\Delta)D - Dt(x+\Delta) + d \tag{3.4}$$

where $X(x) = \delta A(x)$, $t(x) = \delta T(x)$ and $d = \delta D$.

The general solution of (3.4) is

$$t(x) = Q - \frac{1}{2\pi i}\sum_{i,j}\int \mathrm{d}z\ \frac{1}{D_i}w(x-z,q_{ij})X_{ij}(z)E_{ij}. \tag{3.5}$$

Here Q is some constant diagonal matrix and the function $w(x,0)$ should be understood as

$$w(x,0) = \lim_{\varepsilon\to 0}\left(w(x,\varepsilon) - \frac{\mathrm{i}}{1-e^{-\mathrm{i}\varepsilon}}\right) = \zeta(x-\Delta) - \frac{\zeta(\pi)}{\pi}(x-\Delta) - \frac{\mathrm{i}}{2}.$$

Note that these functions solve the equations

$$\frac{1}{2\pi\mathrm{i}}(w(x+\Delta,q_{ij}) - e^{-\mathrm{i}q_{ij}}w(x-\Delta,q_{ij})) = \delta(x) - \frac{1}{2\pi}\delta_{ij}.$$

The solution $t(x)$ obeying the condition $t(0)e = 0$ has the following form:

$$t(x) = \frac{1}{2\pi\mathrm{i}}\sum_{i,j}\int \mathrm{d}z\ \left(\frac{1}{D_i}w(-z,q_{ij})X_{ij}(z)E_{ii} - \frac{1}{D_i}w(x-z,q_{ij})X_{ij}(z)E_{ij}\right). \tag{3.6}$$

Performing the variation of (3.6) with respect to $X_{mn}(z)$ one obtains

$$\left.\frac{\delta T_{kl}(x)}{\delta A_{mn}(z)}\right|_{\mathcal{P}_{\rm red}} \equiv Q^{kl}_{mn}(x+\Delta, z) = \frac{1}{2\pi \mathrm{i}}\frac{1}{D_k}\,(w(-z, q_{kn})\delta_{kl}\delta_{km} - w(x-z, q_{kl})\delta_{km}\delta_{ln}).$$

Thus on the reduced space we obtain

$$\frac{1}{\gamma}\{C_1(x), T_2(y-\Delta)\}|_{\rm red} = \kappa_{12}(x,y)L_1(x) - L_1(x)\omega_{12}(x,y)$$

where

$$\kappa_{12}(x,y) = \mathrm{tr}_3\int \mathrm{d}z\ (D_3 r^{13}_+(x-z-2\Delta) - r^{13}_+(x-z)D_3)Q_{23}(y,z)$$

$$\omega_{12}(x,y) = \mathrm{tr}_3\int \mathrm{d}z\ (D_3 r^{13}_+(x-z-2\Delta) - r^{13}_-(x-z)D_3)Q_{23}(y,z).$$

We also obtain

$$\frac{1}{\gamma}\{T_1(x-\Delta), C_2(y)\} = -P\kappa_{12}(y,x)PL_2(y) + L_2(y)P\omega_{12}(y,x)P.$$

By using the relation

$$D_j Q^{kl}_{ij}(x,z) - D_i Q^{kl}_{ij}(x, z-2\Delta) = \delta(x-z)\delta_{ik}\delta_{jl} - \delta(z-\Delta)\delta_{ik}\delta_{kl} \equiv S^{kl}_{ij}$$

we find

$$\kappa_{ij\ kl}(x,y) = -r_+(x-y)_{ij\ kl} + \sum_m r_+(x-\Delta)_{ij\ km}\delta_{kl}$$

$$\omega_{ij\ kl}(x,y) = k_{ij\ kl}(x,y) + 2\pi D_i Q^{kl}_{ji}(y,x).$$

Recall that

$$\frac{1}{\gamma}\{C_1(x), C_2(y)\} = -r_\mp(x-y)C_1(x)C_2(y) - C_1(x)C_2(y)r_\pm(x-y)$$
$$+ C_2(y)r_+(x-y)C_1(x) + C_1(x)r_-(x-y)C_2(y).$$

Substituting $\{C, T\}$, $\{T, C\}$ and $\{C, C\}$ brackets into (3.3) we can rewrite the $\{\mathcal{L}, \mathcal{L}\}$ bracket in the following form:

$$\left.\frac{1}{\gamma}\{\mathcal{L}_1(x), \mathcal{L}_2(y)\}\right|_{\rm red} = -L_1(x)L_2(y)k^+(x,y) - k^-(x,y)L_1(x)L_2(y)$$
$$+ L_1(x)s^-(x,y)L_2(y) + L_2(y)s^+(x,y)L_1(x) \tag{3.7}$$

where

$$k^-(x,y) = r_-(x-y) + \kappa_{12}(x,y) - P\kappa_{12}(y,x)P - \{T_1(x-\Delta), T_2(y-\Delta)\}$$

$$k^+(x,y) = r_+(x-y) + \omega_{12}(x,y) - P\omega_{12}(y,x)P - \{T_1(x-\Delta), T_2(y-\Delta)\}$$

$$s^-(x,y) = r_-(x-y) + \omega_{12}(x,y) - P\kappa_{12}(y,x)P - \{T_1(x-\Delta), T_2(y-\Delta)\}$$

$$s^+(x,y) = r_+(x-y) + \kappa_{12}(x,y) - P\omega_{12}(y,x)P - \{T_1(x-\Delta), T_2(y-\Delta)\}.$$

It is easy to find $Pk^\pm(x,y)P = -P\delta(x-y) - k^\pm(y,x)$ and $Ps^\pm(x,y)P = \pm s^\mp(y,x)$. We also have one more important identity:

$$k^+(x,y) + k^-(x,y) = s^+(x,y) + s^-(x,y).$$

To complete the calculation we should find the bracket $\{T_{ij}(x-\Delta), T_{kl}(y-\Delta)\}$ on the reduced space. The straightforward manipulations lead to a divergent result. By this reason we define this bracket as follows:

$$\{T_{ij}(x-\Delta), T_{kl}(y-\Delta)\} = \tfrac{1}{2}\lim_{\varepsilon\to 0}\left(\{T_{ij}(x-\Delta), T^{\varepsilon}_{kl}(y-\Delta)\} + \{T^{\varepsilon}_{ij}(x-\Delta), T_{kl}(y-\Delta)\}\right)$$

where $T^{\varepsilon}_{kl}(x)$ is defined as a solution of the factorization problem with the boundary condition $T(\varepsilon)e = e$. We have

$$\begin{aligned}\{T_{ij}(x-\Delta), T^{\varepsilon}_{kl}(y-\Delta)\} &= \int \mathrm{d}z\ \mathrm{d}z'\ Q^{ij}_{mn}(x,z)Q^{kl\ \varepsilon}_{sp}(y,z')\{A_{mn}(z), A_{sp}(z')\}\\ &= \gamma\int \mathrm{d}z\ \mathrm{d}z'\ \big(-r_+(z-z')_{mn\ sp}(D_n Q^{ij}_{mn}(x,z)\\ &\quad - D_m Q^{ij}_{mn}(x,z-2\Delta))S^{kl\ \varepsilon}_{sp}(y,z')\\ &\quad - 2\pi P_{mn\ sp}\delta(z-z'+2\Delta)D_m Q^{ij}_{mn}(x,z)S^{kl\ \varepsilon}_{sp}(y,z')\big).\end{aligned}$$

One can prove the cancellation of the singularities as $\varepsilon\to 0$. The result for the bracket $\{T, T\}$ is

$$\begin{aligned}\frac{1}{\gamma}\{T_{ij}(x-\Delta), T_{kl}(y-\Delta)\} &= -r_+(x-y)_{ij\ kl} + \sum_m r_+(x-\Delta)_{ij\ km}\delta_{kl} + \sum_m r_-(\Delta-y)_{im\ kl}\delta_{ij}\\ &\quad + \frac{1}{\mathrm{i}}w(x-y+\Delta, q_{ik})\delta_{jk}\delta_{il} - \frac{1}{\mathrm{i}}w(x, q_{ik})\delta_{jk}\delta_{kl} + \frac{1}{\mathrm{i}}w(y, q_{ki})\delta_{ij}\delta_{il}\\ &\quad + \frac{1}{2}\delta_{ij}\delta_{ik}\delta_{il} + \frac{1}{\mathrm{i}}\left(\zeta(q_{ik}) - \frac{\zeta(\pi)}{\pi}q_{ik}\right)\delta_{ij}\delta_{kl}(1-\delta_{ik})\\ &\quad - \frac{1}{2}\sum_{a<b}(E_{ab}-E_{ba})_{ik}\delta_{ij}\delta_{kl}.\end{aligned}$$

Combining all the pieces together and taking into account the identity $e^{-\mathrm{i}s}w(x,s) = -w(-x,-s)$ we obtain the following expression for the coefficients:

$$\begin{aligned}k^{-}_{ij\ kl}(x,y) &= -\frac{1}{\mathrm{i}}\zeta(q_{ik})\delta_{ij}\delta_{kl}(1-\delta_{ik}) - \frac{1}{\mathrm{i}}\left(\zeta(x-y)+\zeta(y-\Delta)-\zeta(x-\Delta)\right)\delta_{ij}\delta_{ik}\delta_{il}\\ &\quad - \frac{1}{\mathrm{i}}\left(w(x-y+\Delta, q_{ik})\delta_{il}\delta_{jk} + w(y, q_{ki})\delta_{il}\delta_{ij} - w(x, q_{ik})\delta_{jk}\delta_{kl}\right)(1-\delta_{ik})\\ &\quad + \frac{1}{\mathrm{i}}\frac{\zeta(\pi)}{\pi}q_{ik}\delta_{ij}\delta_{kl} + \frac{1}{2}\sum_{a<b}(E_{ab}-E_{ba})_{ik}\delta_{ij}\delta_{kl}\end{aligned}$$

$$\begin{aligned}k^{+}_{ij\ kl}(x,y) &= \frac{1}{\mathrm{i}}\left(\zeta(x-y) - \frac{\zeta(\pi)}{\pi}(x-y)\right)\delta_{ij}\delta_{ik}\delta_{il}\\ &\quad + \frac{1}{i}\left(w(x-y+\Delta, q_{ik})\delta_{jk}\delta_{il} - \zeta(q_{ik})\delta_{ij}\delta_{kl}\right)(1-\delta_{ik})\end{aligned}$$

$$+\frac{1}{\mathrm{i}}\frac{\zeta(\pi)}{\pi}q_{ik}\delta_{ij}\delta_{kl}+\frac{1}{2}\sum_{a<b}(E_{ab}-E_{ba})_{ik}\delta_{ij}\delta_{kl}$$

$$s^{-}_{ij\ kl}(x,y)=-\frac{1}{\mathrm{i}}\left(\zeta(y-\Delta)-\frac{\zeta(\pi)}{\pi}(y-\Delta)\right)\delta_{ij}\delta_{ik}\delta_{il}$$

$$-\frac{1}{\mathrm{i}}\left(w(y,q_{ki})\delta_{ij}\delta_{il}+\zeta(q_{ik})\delta_{ij}\delta_{kl}\right)(1-\delta_{ik})$$

$$+\frac{1}{i}\frac{\zeta(\pi)}{\pi}q_{ik}\delta_{ij}\delta_{kl}+\frac{1}{2}\sum_{a<b}(E_{ab}-E_{ba})_{ik}\delta_{ij}\delta_{kl}$$

$$s^{+}_{ij\ kl}(x,y)=\frac{1}{\mathrm{i}}\left(\zeta(x-\Delta)-\frac{\zeta(\pi)}{\pi}(x-\Delta)\right)\delta_{ij}\delta_{ik}\delta_{il}$$

$$+\frac{1}{\mathrm{i}}\left(w(x,q_{ik})\delta_{jk}\delta_{kl}-\zeta(q_{ik})\delta_{ij}\delta_{kl}\right)(1-\delta_{ik})$$

$$+\frac{1}{\mathrm{i}}\frac{\zeta(\pi)}{\pi}q_{ik}\delta_{ij}\delta_{kl}+\frac{1}{2}\sum_{a<b}(E_{ab}-E_{ba})_{ik}\delta_{ij}\delta_{kl}.$$

It is instructive to note that one can check by direct calculation that the term $\zeta(\pi)q_{ik}\delta_{ij}\delta_{kl}/\mathrm{i}\pi$ in the expressions obtained for the k's and the s's does not contribute to the bracket $\{\mathcal{L},\mathcal{L}\}$.

Recall (see equation (2.9)) that

$$L_{ii}(x)=\frac{1}{\sqrt{N}}w(x,h)b_i \tag{3.8}$$

so to obtain the bracket $\{b_i,b_j\}$ it is sufficient to examine only the $\{L_{ii},L_{jj}\}$ bracket. The crucial point which can be checked by the direct calculation is that the bracket of $\mathcal{L}_{ii}$ with the constraint (2.2) vanishes on $\mathcal{P}_{\mathrm{red}}$ in the limit $\varepsilon\to 0$. Thus, there is no contribution from the Dirac term to the $\{L_{ii},L_{jj}\}$ bracket.

By substituting the expressions obtained above for k and s in equation (3.7), one obtains for $i\neq j$

$$\frac{1}{\gamma}\{L_{ii}(x),L_{jj}(y)\}=\frac{1}{\mathrm{i}}L_{ji}(x)L_{ij}(y)w(x-y+\Delta,q_{ij})-\frac{1}{\mathrm{i}}L_{ij}(x)L_{ji}(y)w(x-y+\Delta,q_{ji}). \tag{3.9}$$

It follows from this equation that

$$\frac{\mathrm{i}}{\gamma}\{b_i,b_j\}=b_ib_j\frac{w(x,s_{ji})w(y,s_{ij})w(x-y+\Delta,q_{ij})-w(x,s_{ij})w(y,s_{ji})w(x-y+\Delta,q_{ji})}{w(x,h)w(y,h)}. \tag{3.10}$$

By using one of the known elliptic identities [8]†, we obtain

$$\frac{\mathrm{i}}{\gamma}\{b_i,b_j\}=b_ib_j\left(2\zeta(q_{ij})-\zeta(q_{ij}+h)-\zeta(q_{ij}-h)\right). \tag{3.11}$$

To complete the examination of the Poisson structure on the reduced phase space one should find the bracket $\{\mathcal{L},D\}$ and $\{D,D\}$. Performing the straightforward but rather tedious

† It is interesting to note that this identity can easily be obtained from the x, y-independence condition for the right-hand side of (3.10).

calculations following the same line as above, we find

$$\{D[A]_1, D[A]_2\}|_{\text{red}} = 0 \tag{3.12}$$

$$\frac{1}{\gamma}\{\mathcal{L}(x)_1, D[A]_2\}|_{\text{red}} = -\sum_{i,j} L_{ij}(x) D_j E_{ij} \otimes E_{jj}. \tag{3.13}$$

It is worthwhile to point out that there are no Dirac terms in these brackets because $D[A]$ is invariant with respect to the action of the whole affine group $\widetilde{GL(N)}$.

Now for the convenience of the reader, we list the Poisson brackets obtained in terms of the coordinates on $\mathcal{P}_{\text{red}}$:

$$\begin{aligned} &\{q_i, q_j\} = 0 \\ &\frac{\mathrm{i}}{\gamma}\{q_i, b_j\} = b_j \delta_{ij} \\ &\frac{\mathrm{i}}{\gamma}\{b_i, b_j\} = b_i b_j \left(2\zeta(q_{ij}) - \zeta(q_{ij} + h) - \zeta(q_{ij} - h)\right). \end{aligned} \tag{3.14}$$

One can see that the dynamical system defined by (3.14) and (2.12) is simply the elliptic Ruijsenaars–Schneider model.

In [7] the dynamical r-matrix for L^{Ruij} was obtained by direct calculation with the help of the Poisson structure (3.14). Comparing the r-matrix coefficients k and s with those in [7] we see that, in fact, they differ by the tensor $\frac{1}{2}\sum_{a<b}(E_{ab} - E_{ba})_{ik}\delta_{ij}\delta_{kl}$. However, in our calculations of the bracket $\{\mathcal{L}, \mathcal{L}\}$ we ignored the contribution from the Dirac term. We conjecture that the Dirac term is alone responsible for cancelling this tensor.

4. Equations of motion

The equations of motion for the Hamiltonian (2.12) are given by

$$\dot{D} = \{\mathrm{tr}\, L(x), D\} = -\gamma L(x)_{\text{diag}}\; D \tag{4.1}$$

and

$$\dot{L}(y) = \{\mathrm{tr}\, L(x), L(y)\} = [L(y), M(x, y)] \tag{4.2}$$

where

$$M(x, y) = -\gamma \mathrm{i} \sum_{kl} \left(w(x, -q_{kl}) L(x)_{kl} E_{kk} - w(x - y + \Delta, -q_{kl}) L(x)_{kl} E_{kl}\right). \tag{4.3}$$

Here we use equation (3.7) and the explicit form of k and s. Since $\mathrm{tr}\, L(x)$ is invariant function the contribution from the Dirac term vanishes. For the the convenience of the reader we note that by using the elliptic function identities [8] one can rewrite $M \equiv M(x+\Delta, y+\Delta)$ in the following form:

$$M = \frac{\gamma}{i} l(x, h)\Bigg(\frac{\zeta(x+h) - \zeta(x-y)}{l(y, h)} L(y+\Delta) - (\zeta(x+h) - \zeta(x))\Bigg(\sum_i b_i\Bigg) I \tag{4.4}$$

$$+ \sum_k E_{kk} \sum_{i\neq k} (\zeta(q_{ik}) - \zeta(q_{ik} - h)) b_i - \frac{\zeta(\pi)}{\pi} \sum_k b_k E_{kk} \tag{4.5}$$

$$+ \sum_{k\neq l} \frac{\zeta(q_{kl}) - \zeta(q_{kl} + y + h)}{l(y, h)} L_{kl}(y+\Delta) E_{kl}\Bigg) \tag{4.6}$$

where we have introduced $l(x,h) = w(x+\Delta,h)$. The first two terms in (4.4) are irrelavant, so M coincides with the standard M-matrix of the elliptic RS system.

We show that the general solution of the equations of motion for the elliptic Ruijsenaars–Shneider model is given by

$$D(t) = D\left[e^{-2\pi\gamma L_0(x)t} D_0\right] \tag{4.7}$$

where $D \equiv D[A]$ denotes the solution of the factoriztion problem (3.1):

$$A(x) = T(x-\Delta)D[A]T(x+\Delta)^{-1} \tag{4.8}$$

and D_0, and $L_0(x)$ are the coordinates and the L-operator at $t=0$, respectively.

To prove (4.7) we start with calculating the derivative $\dot{D}(t)$:

$$\dot{D}(t) = \int \mathrm{d}z \left.\frac{\delta D[A]}{\delta A_{ij}(z)}\right|_{A=A_t} \frac{\mathrm{d}(A_t)_{ij}(z)}{\mathrm{d}t} \tag{4.9}$$

where $A_t(x) = e^{-2\pi\gamma L_0(x)t} D_0$. One can find the derivative $\delta D[A]/\delta A_{ij}(z)$ by performing the variation of (4.8):

$$(T^{-1}(x-\Delta)\delta T(x-\Delta))D[A] - D[A](T^{-1}(x+\Delta)\delta T(x+\Delta)) + \delta D = X(x) \tag{4.10}$$

where the notation $X(x) = T^{-1}(x-\Delta)\delta A(x)T(x+\Delta)$ was introduced. In contrast to (3.4), in equation (4.10) we do not impose the constraint $T=1$.

Now we solve (4.10) for δD:

$$\delta D = \int \frac{\mathrm{d}x}{2\pi} X(x)_{kk} E_{kk}. \tag{4.11}$$

Equation (4.10) also allows one to find the matrix

$$T^{-1}(x)\delta T(x) = \sum_{k,l} \int \frac{\mathrm{d}z}{2\pi\mathrm{i}} \left(\frac{1}{D_k} w(-z,q_{kl})X(z)_{kl}E_{kk} - \frac{1}{D_k} w(x-z,q_{kl})X(z)_{kl}E_{kl}\right) \tag{4.12}$$

which will be used in what follows. From equations (4.11), (4.12) we find

$$\frac{\delta D[A]_{kk}}{\delta A_{ij}(z)} = \frac{1}{2\pi} T^{-1}_{ki}(z-\Delta)T_{jk}(z+\Delta) \tag{4.13}$$

and

$$\left(T^{-1}(x)\frac{\delta T(x)}{\delta A_{ij}(z)}\right)_{kl} = \frac{\delta_{kl}}{2\pi\mathrm{i}} \sum_s \frac{w(-z,q_{ks})}{D_k} T^{-1}_{ki}(z-\Delta)T_{js}(z+\Delta)$$
$$- \frac{1}{2\pi\mathrm{i}} \frac{w(x-z,q_{kl})}{D_k} T^{-1}_{ki}(z-\Delta)T_{jl}(z+\Delta). \tag{4.14}$$

Substituting equation (4.14) in (4.9) and taking into account $\dot{A}_t(x) = -2\pi\gamma L_0(x)A_t(x)$, we obtain

$$\dot{D}(t)_{kk} = -\gamma \int \mathrm{d}z\ T^{-1}_{ki}(z-\Delta)L_0(z)_{im}T_{mn}(z-\Delta)T^{-1}_{ns}(z-\Delta)A_t(z)_{sj}T_{jk}(z+\Delta) \tag{4.15}$$

which, with the help of (4.8), reads as follows:

$$\dot{D}(t) = -\gamma \int \mathrm{d}z\ (T^{-1}(z-\Delta)L_0(z)T(z-\Delta))_{\mathrm{diag}}\ D(t). \tag{4.16}$$

The last formula implies the notation

$$\hat{L}_t(x) = T^{-1}(x-\Delta)(t)L_0(x)T(x-\Delta)(t) \tag{4.17}$$

which provides the Lax representation $\mathrm{d}\hat{L}_t(x)/\mathrm{d}t = [\hat{L}_t(x), \hat{M}(x)]$ with $\hat{M}(x) = T^{-1}(x-\Delta)\dot{T}(x-\Delta)$.

Let us show that the Lax operator $\hat{L}_t(x)$ coincides with the L-operator of the elliptic Ruijsenaars–Schneider model. To this end we calculate $\hat{M}(x)$ explicitly. We have

$$\hat{M}_{kl}(x) = \int \left(T^{-1}(x-\Delta)\frac{\delta T(x-\Delta)}{\delta A_{ij}(z)}\right)_{kl}\Bigg|_{A=A_t} \frac{\mathrm{d}A_t(z)_{ij}}{\mathrm{d}t}.$$

Substituting equation (4.14) and using the relation $e^{-\mathrm{i}s}w(x,s) = -w(-x,-s)$ we obtain

$$\hat{M}(x) = -\gamma \mathrm{i}\int \mathrm{d}z \sum_{kl}\Big(w(z,-q_{kl})\hat{L}_t(z)_{kl}E_{kk} - w(z-x+\Delta,-q_{kl})\hat{L}_t(z)_{kl}E_{kl}\Big). \tag{4.18}$$

Note that this expression literally coincides with (4.3) if we change $\hat{L}_t$ for L. Since at $t=0$ the operators $\hat{L}$ and L are equal to L_0, they coincide for any t.

5. Conclusion

We have proved that the elliptic Ruijsenaars–Schneider model can be obtained by means of a reduction procedure. It is worth pointing out that we have used not the Hamiltonian, but rather the Poisson reduction technique. Our construction is specified by the choice of the trigonometric r-matrix for the Poisson structure on the HD and by fixing the special value of the momentum map. By varying the right-hand side of the momentum map equation one can derive some other systems. For instance, it is not difficult to specify the momentum map equation in a way that leads to the elliptic Calogero–Moser model. It clarifies the coincidence of the dynamical r-matrices for these two models pointed out in [7].

We have considered the simplest example of the HD for $\widetilde{GL(N)}$. It seems to be interesting to examine the Poissonian reductions of the HD that correspond to some other choices of Lie groups or r-matrices.

Acknowledgments

The authors are grateful to L Chekhov, A Gorsky and N A Slavnov for valuable discussions. This work is supported in part by the RFFR grants N96-01-00608 and N96-01-00551 and by the ISF grant a96-1516.

References

[1] Sklyanin E K 1994 *Algebra i Analiz* **6** 227
[2] Avan J, Babelon O and Talon M 1994 *Algebra i Analiz* 67
[3] Braden H W and Suzuki T 1994 *Lett. Math. Phys.* **30** 147
[4] Billey E, Avan J and Babelon O 1994 *Phys. Lett.* **188A** 263
[5] Avan J and Rollet G 1995 The classical r-matrix for the relativistic Ruijsenaars–Schneider system *Preprint* BROWN-HET-1014
[6] Arutyunov G E and Medvedev P B 1995 Generating equation for r-matrices related to dynamical systems of Calogero type *Preprint* hep-th/9511070 (*Phys. Lett.* A to appear)
[7] Suris Yu B 1996 Why are the rational and hyperbolic Ruijsenaars–Schneider hierarchies governed by the same R-matrix as the Calogero–Moser ones? *Preprint* hep-th/9602160

Suris Yu B 1996 Elliptic Ruijsenaars–Schneider and Calogero–Moser hierarchies are governed by the same r-matrix *Preprint* solv-int/9603011

[8] Nijhoff F W, Kuznetsov V B, Sklyanin E K and Ragnisco O 1996 *J. Phys. A: Math. Gen.* **29** L333–40

[9] Avan J and Talon M 1993 *Phys. Lett.* **303B** 33–7

[10] Arnol'd V I 1989 *Mathematical Methods of Classical Mechanics (Springer Graduate Texts in Mathematics 60)* (Berlin: Springer)

[11] Olshanetsky M A and Perelomov A M 1976 *Invent. Math.* **37** 93

[12] Kazhdan D, Kostant B and Sternberg S 1978 *Commun. Pure Appl. Math.* **31** 481

[13] Olshanetsky M A and Perelomov A M 1981 *Phys. Rep.* **71** 313

[14] Olshanetsky M A and Perelomov A M 1983 *Phys. Rep.* **94** 6

[15] Ruijsenaars S N 1987 *Commun. Math. Phys.* **110** 191

[16] Semenov-Tian-Shansky M A 1992 *Teor. Mat. Fiz.* **93** 302 (in Russian)

[17] Gorsky A and Nekrasov N 1994 *Nucl. Phys.* B **414** 213
Gorsky A and Nekrasov N 1995 *Nucl. Phys.* B **436** 582
Gorsky A 1994 Integrable many body systems in the field theories *Preprint* UUITP-16/94

[18] Gorsky A and Nekrasov N 1994 Elliptic Calogero–Moser system from two-dimensional current algebra *Preprint* hep-th/9401021

[19] Reshetikhin N Yu and Semenov-Tian-Shansky M A 1990 *Lett. Math. Phys.* **19** 133–42